

NIM ON HYPERCUBES

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ABSTRACT

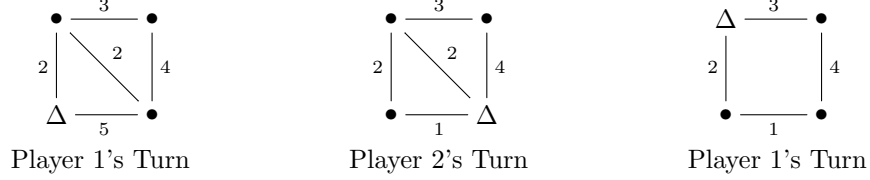
The ordinary game of Nim has a long history and is well-known in the area of combinatorial game theory. The solution to the ordinary game of Nim has been known for many years and lends itself to numerous other solutions to combinatorial games. Nim was extended to graphs by taking a fixed graph with a playing piece on a given vertex and assigning positive integer weight to the edges that correspond to a pile of stones in the ordinary game of Nim. Players move alternately from the playing piece across incident edges, removing weight from edges as they move. This paper solves Nim on hypercubes in the unit weight case completely. We briefly discuss the arbitrary weight case and its ties to known results.

1. BACKGROUND

The graphs we will consider are finite and undirected with no multiple edges or loops. We will often want to label the vertices and edges. When we do, the edge between vertex v_i and v_j will be denoted e_{ij} . Additional graph theory terminology, including path, vertex degree, and graph isomorphism, will be assumed as found in [1]. When we refer to the length of a cycle or path, we will call it even or odd by the number of edges the cycle or path contains.

1.1. How to Play. To play Nim on graphs, two players first agree on a finite, undirected, integrally weighted graph and a fixed starting position. The position of the game is indicated by a positional piece which we will denote by Δ . The game starts with P_1 choosing an edge incident with Δ to move across. As a player moves across an edge, the player must lower the weight of the edge by a positive integer amount. The positional piece Δ moves with the move of the player so that when a player comes to rest on the other vertex incident with that edge, the next player must start with that vertex and move across edges incident with the new position of Δ . If either player lowers the weight of an edge to zero, the edge is no longer playable. For ease of notation, we will delete the edge from the picture of the graph if the weight is decreased to zero (see Figure 1). Play continues in this back-and-forth fashion until a player can no longer move since there are no edges incident with Δ .

FIGURE 1. An example of the first two moves in a game of Nim on graphs.



1.2. Nim on Graphs Definitions.

Definition 1.1. Given a graph G with edge set $E(G)$ and vertex set $V(G)$ we will call the non-negative integer value assigned to each $e \in E(G)$ the **weight** of the edge and denote the weight of edge e_{ij} by $\omega(e_{ij})$.

When we say a graph has *unit weight*, we precisely mean that $\omega(e) = 1$ for all $e \in E(G)$. We will often discuss *uniformly weighted* graphs, meaning that $\omega(e) = k$ for all $e \in E(G)$ and for some $k \in \mathbb{Z}^+$.

For any graph G we assume $\omega(e_{ij}) \neq 0$ for all $e_{ij} \in E(G)$ at the start of a game. When an edge is such that $\omega(e) = 0$ we will delete it from the graph entirely, since it is no longer a playable edge. Given a game graph G with weight assignment $\omega_G(e)$, denote by P_1 the first player to move from the starting vertex, and denote by P_2 the player to move after P_1 . The indicator piece Δ denotes the vertex from which a player is to move. We will always enumerate vertices in such a way that Δ is on v_1 at the start of a game.

Definition 1.2. For either player and from a given position Δ on vertex v_j , we define the set of vertices to which a player may legally move to from Δ to be the **option** of the player. The set of options of player i at vertex v_j will be denoted by $O(P_i, v_j)$.

Certainly for a vertex to exist in the set of options the incident edge must be adjacent to Δ . Thus $O(P_i, v_j) = \{v_k \in V(G) : \Delta = v_j; e_{jk} \in E(G); \omega(e_{jk}) \neq 0\}$. We will omit v_j when the position of Δ is apparent.

Definition 1.3. For either player and from a given position Δ on vertex v_j , we call the decision of how much weight to remove from an edge e_{ji} the **choice** of the player.

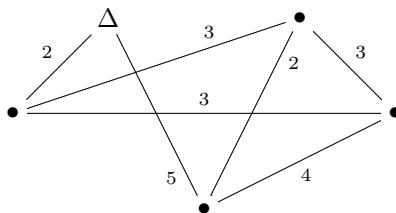
Thus for any given option with $\omega(e_{ij}) > 1$, the player has a choice of whether or not to remove all weight, or exactly how much weight to remove.

Definition 1.4. We will say that a pair of P_i 's options are **isomorphic** if given two options, $v_j, v_k \in O(P_i, v_i)$, there exists a graph isomorphism between v_j and its neighbors and v_k and its neighbors. We will say that two options are **identical** if in addition to being isomorphic, the subgraph induced by v_i and each $v_j \in O(P_i, v_i)$ have the same weight assignment.

We will use the word *option* exclusively when we are referring to the vertex a player will move to, and the word *choice* to refer to the amount of weight across the option's edge to be removed during play. Hence during any given move, a player will have the option of which vertex to move to, and the choice of how much weight to remove.

Notice that the definition of isomorphic requires that the vertices in the set of options have the same degree, and that there is a bijection between the options of the vertices within the set of isomorphic options (see Figure 2). In other words, if for all $v_j, v_k \in O(P_i, v_i)$ we have that $O(P_i, v_j) \cong O(P_i, v_k)$ then the options of v_i are isomorphic. We will also talk about graphs being isomorphic within the context of Nim on graphs. This will be necessary to cut down on cases to consider within games.

FIGURE 2. The options at Δ are isomorphic but not identical.

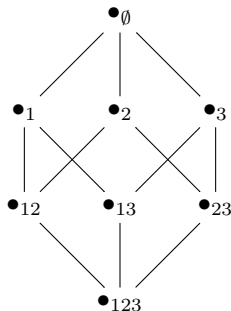


2. NIM ON THE HYPERCUBE WITH UNIT WEIGHT

Definition 2.1. The n -dimensional hypercube, or the n -cube, Q_n is the graph K_2 if $n = 1$, while for $n \geq 2$, Q_n is defined recursively as $Q_{n-1} \times K_2$ [1].

We can also think of the n -cube as the graph whose vertices are labeled by the binary n -tuples (a_1, a_2, \dots, a_n) where each a_i is either 0 or 1 for $1 \leq i \leq n$ and such that two vertices are adjacent if and only if their corresponding n -tuples differ at precisely one coordinate. This is the view of hypercubes that we will adopt in what follows, along with the following alternate labeling. Label each vertex $a = (a_1, a_2, \dots, a_n)$ of the hypercube Q_n by the corresponding set $X_a = \{i : a_i = 1\}$ [2]. Then we will draw the Q_n in the plane so that the vertical coordinates of the vertices are in order by the size of the sets labeling them (see Figure 3). We will call this the level labeling scheme and use it throughout the hypercube section.

FIGURE 3. Here is the Q_3 with the level labeling scheme.



Definition 2.2. *The parity of a vertex in Q_n is the parity of the number of 1's in its name, even or odd [2].*

This implies that each edge of the Q_n has an even vertex and an odd vertex as endpoints (see Figure 3). This means that the even vertices form an independent set, as do the odd vertices. Hence Q_n is bipartite for any n [2].

Since we typically start with Δ on the lowest numerically denoted vertex. Here we will start with Δ on vertex \emptyset . With this level labeling scheme, we can think of the vertices at different levels corresponding to the number of digits in the vertex labels. Thus in the example of the Q_3 we have levels $\emptyset, 1, 2$, and 3 .

Throughout this section we will assume that the weight of each edge of the hypercube has unit weight.

Lemma 2.3. *P_1 can keep game play on the Q_{2n+1} within the confines of levels $\emptyset, 1$, and 2 .*

Proof. Let Q_{2n+1} have unit weight and label each vertex by the X_a scheme described above so that \emptyset is at vertex $(0, 0, \dots, 0)$. Give the Q_{2n+1} the level labeling scheme. Assume that Δ starts at vertex \emptyset . Note that since the Q_{2n+1} is regular of order $2n + 1$ any choice of starting vertex is isomorphic.

Every hypercube is bipartite. Thus we can observe that P_1 's vertices all have even parity, and P_2 's vertices have odd parity according to the labeling scheme.

Suppose P_1 is at vertex ij in level 2. Since we want to show that P_1 can opt not to move down to level 3, we will show that there is always an option in level 1 for any ij in level 2. Since P_1 is playing from vertex ij , either P_2 moved from i or from j in level 1. Without loss of generality, assume P_2 moved from i so that $e_{i,ij}$ is no longer an option for P_1 .

By way of contradiction, suppose that P_1 cannot move to j from ij . This implies that $e_{j,ij}$ has been used already. This can only occur in one of two ways: the first case is that P_1 moved to j via $e_{j,ij}$ on a previous move, and the second case is that P_2 moved from j to ij via $e_{j,ij}$ on a previous move.

In the first case, if P_1 moved from ij to j then it must be the case that P_2 was on level three and moved from some ijk to ij . This is because we are assuming that just now P_2 moved from i to ij and thus could not have made that move previously. (Recall that since we have unit weight, once an edge has been moved across once it is no longer a playable edge.) This contradicts the fact that P_1 would not make such a move unless forced to. Clearly P_1 was not forced to previously move down to level 3 since it is only now that a move to vertex i is no longer possible.

In the second case, if P_2 moved from j to ij but P_1 did not move from ij to i since it remains, then P_1 moved down to some ijk , again a contradiction.

Thus P_1 always has a level 1 option and hence can keep P_2 within levels $\emptyset, 1$, and 2 . \square

Theorem 2.4. *Assume $\omega(e) = 1$ for all $e \in Q_{2n+1}$. Then P_1 can win the Q_{2n+1} for any $n \geq 1$.*

Proof. Assume $\omega(e) = 1$ for all $e \in E(Q_{2n+1})$, that $n \in \mathbb{Z}$, and $n \geq 1$. Label the digits according to X_a and the level labeling scheme. Start with Δ on \emptyset .

Since all hypercubes are bipartite, we know that P_1 's vertices have even parity, and P_2 vertices have odd parity. By Lemma 2.3, P_1 can keep P_2 within the confines of levels $\emptyset, 1$, and 2 . Because of this, consider only these three levels. In essence, "chop off" levels 3 through $2n + 1$.

With P_1 at \emptyset at the start, notice that the vertices in level \emptyset and 1 are all odd degree. Since we are considering the graph without levels 3 through $2n + 1$, the vertices in level 2 are all of degree 2. Also, since we are assuming each edge has unit weight, when a player moves across an edge, it is deleted. Thus P_1 starts on an odd degree vertex and P_2 starts on an even degree vertex at each of their respective moves. This implies that P_1 always has an edge to move away from at any vertex (since odd degree implies at least degree 1). However, since Q_{2n+1} is finite, eventually P_2 will come to a vertex of degree 0 and not be able to move.

Thus P_1 always wins the Q_{2n+1} for any positive integer value of n . \square

Theorem 2.5. *Assume that $\omega(e) = 1$. Then P_2 wins the Q_{2n} for all $n \geq 1$.*

Proof. Assume that $n \in \mathbb{Z}$, $n \geq 1$, and $\omega(e) = 1$ for all $e \in E(Q_{2n})$. Label the vertices according to X_a and the level labeling scheme. Start with Δ on \emptyset .

Note that Q_{2n} is regular of degree $2n$, and Q_{2n} is bipartite. Thus P_1 moves from vertices with even parity, and P_2 moves from vertices with odd parity. Also notice that P_1 starts from a vertex of even degree, and each time P_1 moves from \emptyset it is of even degree. Each other vertex is of odd degree when either player moves from it. This is because the degree lowers by one each time a player arrives at the vertex. Thus on the first move, P_1 moves from an even degree vertex to what was an even degree vertex. Since the process of moving to a vertex lowers the degree by one each time because of unit weight of the edges, P_2 starts from a vertex that has odd degree. This is true for each player at each vertex except for P_1 at vertex \emptyset .

If a vertex has odd degree when moving from it, a player is guaranteed to be able to move away from the vertex, since an odd degree vertex implies that the degree is at least 1. Thus the only vertex that a player could possibly get stuck at is the \emptyset vertex. Since P_1 is the only player to move from \emptyset by virtue of Q_{2n} being bipartite, P_1 is the only player who is able to lose.

Therefore, since there are only a finite number of moves, P_2 wins the Q_{2n} for any positive integer value of n . \square

With the previous two theorems, we can formulate the following two corollaries.

Corollary 2.6. *P_1 wins the unit weight hypercube if and only if n is odd.*

Corollary 2.7. *P_2 wins the unit weight hypercube if and only if n is even.*

3. A NOTE ABOUT THE HYPERCUBE WITH ARBITRARY WEIGHT

The unit weight hypercube had a nice parity argument to show the winner. Unfortunately, the hypercube weighted arbitrarily is not so easy to solve. We know very quickly that weight matters with the arbitrarily weighted hypercube. Take for a simple example, $Q_2 = C_4$. By previous work in the even cycle section, we know that the winner of the game is decided by the distances to the lowest weight edge. Hence we can tell at least for the even values of n that the weight of the Q_n will matter in determining the winner of the game.

REFERENCES

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- [2] Douglas B. West. *Introduction to graph theory*. Prentice Hall Inc., Upper Saddle River, NJ, 1996.